Pulse Propagation in Multimode Fibers with Frequency-Dependent Coupling

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Abstract—**Marcuse's time-dependent coupled power equations are rederived so as to include the frequency dependence of the coupling coefficients. For the case of white-noise coupling, the solution is then expressed simply in terms of that for frequency-independent coupling.**

THE FOURIER transform of the electromagnetic field in an optical fiber may be represented as an expansion in the structure's normal modes. The coefficients of this expansion are called mode transfer functions.

Marcuse's derivation of coupled power equations [1] can be carried over step by step to derive an analogous set of equations for the covariances of the mode transfer functions

$$I_{kk}(z;\omega,\omega_1) = \langle I_k(z;\omega+\omega_1)I_k^*(z;\omega)\rangle. \tag{1}$$

When this was done the following perturbation equations were obtained:

$$\frac{dI_{kk}}{dz} + (\alpha_k + j\sigma_k)I_{kk} = \pi^{1/2}\bar{\sigma}^2 D \sum_{m\neq k} \{K_{km_{\omega 1}}K_{km}^* I_{mm} - \tfrac{1}{2}[|K_{km_{\omega 1}}|^2 + |K_{km}|^2]I_{kk}\}. \tag{2}$$

$\Gamma_k(\omega) = \alpha_k + j\beta_k(\omega)$ is the propagation constant of the kth fiber mode in the absence of coupling. The subscript ω_1 denotes evaluation at $(\omega+\omega_1)$. $\sigma_k(\omega,\omega_1)$ is defined by $\beta_{k_{\omega 1}} = \beta_k + \sigma_k$. The coupling coefficients K_{km} have been determined for both the dielectric slab [5] and round dielectric fiber [6] and are independent of distance z along the fiber.

It was first assumed that the physical imperfection responsible for coupling the modes had a Gaussian-shaped correlation function characterized by parameters $\bar{\sigma}$ and D, respectively, the rms amplitude and correlation length of the imperfection [1]. Equation (2) was derived by assuming D sufficiently small

$$|(\beta_k - \beta_l)D|_{\max} \ll 1. \tag{3}$$

Equivalent to the assumption of white-noise coupling, (3) simplifies the calculations considerably.

Subject to (3), Marcuse's power coupling coefficients become [5]

$$h_{km} = \pi^{1/2}\bar{\sigma}^2 D |K_{km}|^2 \equiv D_0 |K_{km}|^2. \tag{4}$$

Considering the h_{km} as functions of frequency, the following arithmetic and geometric means are defined [3]:

$$h_{km}{}^a = \tfrac{1}{2}(h_{km} + h_{km_{\omega 1}}) \tag{5}$$

$$h_{km}{}^g = (h_{km}h_{km_{\omega 1}})^{1/2} = D_0 K_{km}^* K_{km_{\omega 1}}. \tag{6}$$

The last identity in (6) follows from (4), (14), and the fact that the K_{km} are purely imaginary.

Using (5) and (6), (2) becomes

$$\frac{dI_{kk}}{dz} = -(\alpha_k + j\sigma_k)I_{kk} + \sum_{m\neq k}\{h_{km}{}^g I_{mm} - h_{km}{}^a I_{kk}\}. \tag{7}$$

It is now assumed that the kth mode is far enough above cutoff and the range of ω_1 is sufficiently small such that

$$\beta_k(\omega_0 + \omega_1) \simeq \beta_{k0} + \beta_{k1}\omega_1, \qquad |\omega_1| \ll \omega_0. \tag{8}$$

It follows from (8) that

$$\sigma_k(\omega,\omega_1) \equiv \beta_k(\omega+\omega_1) - \beta_k(\omega) = \beta_{k1}\omega_1 \tag{9}$$

independent of the optical frequency ω.

Assuming for the moment that a solution to (7) is available, the need now arises to find a relationship between the covariances I_{kk} and some (as yet unspecified) corresponding time-domain statistics. This relationship is provided by [4, eq. (88)]:

$$\langle |a_{k_e}(z;t)|^2\rangle = \int_{-2B}^{2B}\frac{d\omega_1}{2\pi}\exp(j\omega_1 t) \cdot \left\{\int_{\omega_0-B-(\omega_1-|\omega_1|)/2}^{\omega_0+B-(\omega_1+|\omega_1|)/2}\frac{d\omega}{2\pi}4I_{kk}(z;\omega,\omega_1)\right\} \tag{10}$$

where $a_k(z;t)$ is a band-limited version of the kth mode impulse response

$$a_k(z;t) = \mathrm{Re}\left\{2\int_{\omega_0-B}^{\omega_0+B} I_k(z;\omega)\exp(j\omega t)\frac{d\omega}{2\pi}\right\} = \mathrm{Re}\{a_{k_e}(z;t)\exp(j\omega_0 t)\}. \tag{11}$$

The mean-squared envelope of the impulse response $\langle |a_{k_e}|^2\rangle$ is referred to as the kth mode "pulse response" [4] and may be identified with what Marcuse refers to as mode power.

Equation (7) is now written as

$$dI_{kk}/dz = -(\alpha_k + j\sigma_k + Q_k)I_{kk} + \sum_m h_{km}{}^g(I_{mm} - I_{kk}) \tag{12}$$

where

$$Q_k = \sum_{m\neq k}(h_{km}{}^a - h_{km}{}^g). \tag{13}$$

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Assuming a narrow-band excitation,

$$K_{km}(\omega_0 + \omega_1) \simeq K_{km_0} + K_{km_0}'\omega_1$$
$$| \omega_1 K_{km_0}'/K_{km_0} | \ll 1. \tag{14}$$

Subject to (14), it may be shown that [3]

$$h_{km}^a - h_{km}^g \simeq \tfrac{1}{2} D_0 | K_{km_0}' |^2 \omega_1^2 \tag{15}$$

$$h_{km}^g \simeq h_{km_0}. \tag{16}$$

Substituting (15) into (13),

$$Q_k = \{\tfrac{1}{2} D_0 \sum_{m \neq k} | K_{km_0}' |^2\} \omega_1^2 \equiv q_k \omega_1^2. \tag{17}$$

From (9), (12), (16), and (17)

$$\frac{dI_{kk}}{dz} = -(\alpha_k + j\beta_{k1}\omega_1 + q_k\omega_1^2) I_{kk} + \sum_m h_{km_0}(I_{mm} - I_{kk}). \tag{18}$$

Inspection of (17) reveals that $q_k = 0$ for frequency-independent coupling. With the definition

$$p_k(z;t) = \langle | a_{k_e}(z;t) |^2 \rangle \tag{19}$$

for this special case, and applying (10) to (18) with $q_k = 0$, one obtains

$$\{\partial_z + (1/v_k)\partial_t\} p_k(z;t) = -\alpha_k p_k + \sum_m h_{km_0}(p_m - p_k) \tag{20}$$

the time-dependent coupled power equations for frequency-independent coupling [2].

Assuming that the fiber is sufficiently long, Marcuse's perturbation solution to (18) with $q_k = 0$ may be written as

$$I_{kk}(z;\omega,\omega_1) = c(\omega,\omega_1) B_k \exp(-\alpha z). \tag{21}$$

The coefficient $c(\omega,\omega_1)$ is determined by the excitation, B_k is the steady-state mode power distribution, and α is a complex propagation constant common to all modes [2]. It may be noted that the vector B_k is normalized such that $\sum_k B_k^2 = 1$.

It is now assumed that q_k is nonzero, but sufficiently small such that its effect on the solution for I_{kk} may be adequately represented by a first-order perturbation correction to α. It may then be shown that

$$I_{kk}(z;\omega,\omega_1) = c(\omega,\omega_1) B_k \exp(-\alpha z) \exp(-\bar{q} z \omega_1^2) \tag{22}$$

where

$$\bar{q} = \sum_k B_k^2 q_k. \tag{23}$$

It is seen that each q_k is weighted by the square of the corresponding mode power in forming the average $\bar{q}$.

Since (21) and (10) yield the solution to (20), it follows from (22) and (10) that

$$\langle | a_{k_e}(z;t) |^2 \rangle = p_k(z;t) * F^{-1}\{\exp(-\bar{q} z \omega_1^2)\}. \tag{24}$$

The symbol $*$ denotes convolution.

Assuming that the fiber is sufficiently long and that a great deal of pulse broadening takes place in propagation (i.e., that the input pulse width is much smaller than the output pulse width), the pulse response for frequency-independent coupling may be written as [2]

$$p_k(z;t) = (bB_k/\Delta t) \exp(-\alpha_0 z) \exp\left\{-\left[\frac{t - z/v}{\Delta t/2}\right]^2\right\} \tag{25}$$

where α_0 is an attenuation constant common to all the mode powers, v is an average group velocity, $\Delta t = 4(\alpha_2 z)^{1/2}$ is the pulse width, and b is a constant.

It may be shown from (24) that $\langle | a_{k_e} |^2 \rangle$ is obtained by replacing Δt by $(\Delta t)'$ in (25), where

$$(\Delta t)' = 4[(\alpha_2 + \bar{q})z]^{1/2} = [(\Delta t)^2 + 16\bar{q}z]^{1/2}. \tag{26}$$

Nonzero K_{km}' causes the pulse width to be broader than otherwise expected; the pulse widths of the individual modes are still all the same, as for frequency-independent coupling.

The above perturbation solution was compared with the results obtained by Rowe and Young in their exact analysis of the two-mode random waveguide [3]. Using [7], it was found that (24) and [3, eq. (30)], as prescriptions for converting the pulse response for frequency-independent coupling into that for frequency-dependent coupling, are in exact agreement. This is somewhat surprising in view of the fact that (22) results from a perturbation analysis, while [3, eq. (30)] is the result of an exact analysis.

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